

Math 347 Worksheet
Lecture 16: Completeness, Limits and Convergence
October 10, 2018

Consider $S = \{x \in \mathbb{R} \mid x^2 < 2\}$ and let $\alpha = \sup(S)$. We want to prove that $\sup(S) = \alpha$ satisfies $\alpha^2 = 2$.

(i) Prove that for any positive real numbers a, b one has

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Moreover, if $a \neq b$, then the above is a strict inequality.

(ii) Consider $y = \frac{1}{2} \left(\alpha + \frac{2}{\alpha} \right)$, prove that

$$\alpha^2 < \left(\frac{2}{y} \right)^2.$$

(iii) Prove that if a, b are positive real numbers, then $a^2 < b^2$ implies $a < b$.

(iv) Finish the argument from class to show that $\alpha^2 \neq 2$.

1. Prove that for any a, b positive real numbers, there exist $n \in \mathbb{N}$ such that $an > b$.

2. For $a, b \in \mathbb{R}$ is the statement below true or false?

The interval $(a, b]$ contains its supremum and infimum.

3. Suppose that the sequence $(x_n)_{n \geq 0}$ does not converge to zero. Express this as a formal statement.

4. Consider the sequences $x_n = n$ and $y_n = 1/n$. Are they bounded or monotone? Does any of them converge? Check this using the definitions.

5. Consider $\lim a_n < \lim b_n$ for two sequence a_n and b_n that admit limits. Prove that for N sufficiently large $a_N < b_N$.

- I. If $S \subset \mathbb{R}$ then $\alpha = \sup(S)$ if and only if α is an upper bound of S and there exists a sequence in S converging to α .
- II. If a_n and b_n are two nondecreasing sequences, and

$$\lim(a_n - b_n) = 0.$$

Then both sequences converge and have the same limit. Prove that this is not the case if they are not monotone, i.e. nondecreasing.

- III. Proof of the existence of square roots. We want to check that for every $x \geq 0$ a real number, there exists \sqrt{x} a positive real number, such that $(\sqrt{x})^2 = x$.
- a. Prove that if a sequence a_n converges to L , then a_n^2 converges to L^2 .
- b. Prove that if $k \geq 2$, then $a_n = \frac{1}{k^n}$ converges to 0.
- c. Consider the sequences ℓ_n defined as¹

$$\ell_n = \text{largest multiple of } \frac{1}{10^n}, \text{ s. t. } \ell_n^2 \leq x;$$

and

$$r_n = \text{smallest multiple of } \frac{1}{10^n}, \text{ s. t. } r_n^2 \geq x.$$

First argue that both sequences are well-defined. Then use II. and a. and b. to show that both sequences converge to \sqrt{x} .

¹Here, the $\frac{1}{10^n}$ is not a typo, an important part of the exercise is to understand what does it mean for a number y to a multiple of $\frac{1}{n}$ for some $n \geq 1$.